#13 Proof of "+ = Q" thm, delooping of K-theory, and Higher topological K-theory.

<u>THM</u> $(\mathcal{C}, \mathcal{E})$ exact category, there exists an embedding (fully faithful additive functor), $\mathcal{C} \hookrightarrow A$ into an abelian category s.t.

1) $[a \hookrightarrow b \twoheadrightarrow c] \in \mathcal{E}$ iff $0 \to a \to b \to c \to 0$ is a SES in A

2) $\mathcal{C}$ is closed under extension in A.

For $\mathcal{C}$ exact, we define a new category $Ex(\mathcal{C})$ to be the category w/ $Ob(Ex(\mathcal{C})) = \mathcal{E}$.

Let $E, E' \in \mathcal{E}$ $\quad E = [a \hookrightarrow b \twoheadrightarrow c]$ $\quad E' = [a' \hookrightarrow b' \twoheadrightarrow c']$

then $f \in Mor(E, E')$ is an equivalence class of the form

$$\begin{tikzcd}
E' : & a' \arrow[r, hook] & b' \arrow[r, two heads] & c' \\
 & a \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & b' \arrow[u, equal] \arrow[r, two heads] \arrow[d, hook] & c'' \arrow[u, two heads] \arrow[d, hook] \\
E : & a \arrow[r, hook] & b \arrow[r, two heads] & c
\end{tikzcd}$$

equivalence is given by isomorphism of such diagrams.
compositions are given by pull back / push out of admissible epis / monics.

<u>Note</u>: Now we have a morphism $\varphi(f) = [c' \twoheadleftarrow c'' \hookrightarrow c]$ $\in Hom_{Q\mathcal{C}}(c', c)$. Hence $f \mapsto \varphi(f)$ gives a functor

$$\varphi : Ex(\mathcal{C}) \to Q\mathcal{C}$$
$$[a \hookrightarrow b \twoheadrightarrow c] \mapsto c.$$

Denote $\mathcal{E}_c = \varphi^{-1}(c)$ the fibre category of φ over c.

Note: morphism in $\mathcal{E}_c$

$$\begin{array}{ccccc} a' & \hookrightarrow & b' & \twoheadrightarrow & c \\ \alpha \updownarrow & & \updownarrow \beta & & \| \\ a & \hookrightarrow & b & \twoheadrightarrow & c \end{array} \quad (*)$$

$\Rightarrow \alpha, \beta$ necessarily isomorphisms $\mathrm{Hom}_{\mathcal{E}_c}(E', E) = \{(\alpha, \beta) \in \mathrm{Iso}(\mathcal{C}) \times \mathrm{Iso}(\mathcal{C}) \mid (*) \text{ commutes}\}$

$\Rightarrow \mathcal{E}_c$ is a groupoid.

Lem: $\mathcal{C}$ exact, $S = \mathrm{Iso}\,\mathcal{C} \Rightarrow$

1) $\mathcal{E}_c$ is symmetric monoidal, $\exists$ a faithful monoidal functor $[a \mapsto a \oplus c \twoheadrightarrow c]$

2) let 0 be the zero object, then $\eta_0 : S \to \mathcal{E}_0$ is a ~~homotopy~~ homotopy equivalence.

Using this, we can define an action of S on $\mathcal{E}\mathcal{C}$ by

$$a \oplus [a' \hookrightarrow b' \twoheadrightarrow c'] := [a \oplus a' \hookrightarrow a \oplus b' \twoheadrightarrow c'].$$

note that this action preserves the fibre, hence it descends to an action $S \times \mathcal{E}_c \to \mathcal{E}_c$. We have the associated category $S \ltimes \mathcal{E}_c$ and $S^{-1}\mathcal{E}_c$.

Lemma. If $\mathcal{C}$ is split exact, then $S \ltimes \mathcal{E}_c$ is contractible $\forall c \in \mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(Q\mathcal{C})$.

Pf: sketch: $\mathcal{C}$ split exact $\Rightarrow$ every object of $\mathcal{E}_c$ is isomorphic to an image of η_c. $\Rightarrow S \ltimes \mathcal{E}_c$ connected i.e. $\pi_0 = \{1\}$.

$\mathcal{E}_c$ is also symmetric monoidal $\Rightarrow B(S \ltimes \mathcal{E}_c)$ is a group-like H-space $\Rightarrow$ it has homotopy inverse.

Consider $E = [a \hookrightarrow b \twoheadrightarrow c] \in \mathrm{Ob}(S \ltimes \mathcal{E}_c)$ and a diagonal morphism

$$\begin{array}{cccccc} E & & a & \hookrightarrow & b & \twoheadrightarrow & c \\ \delta_E \downarrow & & \downarrow & & \downarrow & & \| \\ E * E & & a \oplus a & \hookrightarrow & b \times_c b & \twoheadrightarrow & c \end{array}$$

which is represented by $(a, \widetilde{\delta_E}) \in \mathrm{Hom}_{S \rtimes \mathcal{E}_c}(E, E * E)$

$$\begin{array}{ccccc} a \oplus a & \hookrightarrow & b \oplus a & \twoheadrightarrow & c \\ \| & & \cong & & \| \\ a \oplus a & \hookrightarrow & b \times_c C & \twoheadrightarrow & C. \end{array}$$

Hence we have a natural transformation $\delta: id_{S \rtimes \mathcal{E}_c} \longrightarrow *$

On $B(S \rtimes \mathcal{E}_c)$, thus δ induces an homotopy between identity and multiplication by 2. Now using the homotopy inverse,

$0 = (-x) + x \simeq -x + 2x = x \Rightarrow 0 \simeq x$. $\square$.

Cor. $\mathcal{C}$ split exact, $\eta_c: S \to \mathcal{E}_c$ induces a homotopy equivalence $S^{-1}S \xrightarrow{\sim} S^{-1}\mathcal{E}_c$.

Pf: Note we have a sequence $S^{-1}S \to S^{-1}\mathcal{E}_c \xrightarrow{\pi} S \rtimes \mathcal{E}_c$

where π is the projection induced by $S \times \mathcal{E}_c \to \mathcal{E}_c$. We can show this is a homotopy fibration sequence. $\square$

Lem. $\forall f: c' \to c$ in $Q\mathcal{C}$, $\exists$ canonical functor $f^*: \mathcal{E}_c \to \mathcal{E}_{c'}$

and morphisms of functors $\eta: f^* \Rightarrow in$

$$\eta: \{\eta_E: f^*(E) \to E\}$$

and $in: \mathcal{E}_c \hookrightarrow \mathcal{E}x(\mathcal{C})$ is the fibre inclusion functor.

Pf: Let $f: [c' \twoheadleftarrow^{p} c'' \xrightarrow{i} c]$, pull back $E = [a \hookrightarrow b \twoheadrightarrow c]$

$\in Ob(\mathcal{E}_c)$ gives $E'' = [a \hookrightarrow b \times_c c'' \twoheadrightarrow c'']$

Now define $a' := \mathrm{Ker}(b \times_c c'' \twoheadrightarrow c'' \twoheadrightarrow c')$.

then define $f^*(E) = [a' \hookrightarrow \underset{b \times_c c''}{\overset{\|}{b'}} \twoheadrightarrow c']$.

Note that this gives

$$\begin{tikzcd} a' \arrow[r, hook] & b' \arrow[r, two heads] & c' \\ a \arrow[u, hook, "\alpha"] \arrow[r, hook] & b' \arrow[u, equal] \arrow[r, two heads] \arrow[d, hook, "\beta"] & c'' \arrow[u, hook] \arrow[d, hook] \\ a \arrow[u, equal] \arrow[r, hook] & b \arrow[r, two heads] & c \end{tikzcd} \in Mor[Ex(\mathcal{C})].$$

$\square$

Cor. $\varphi : Ex(\mathcal{C}) \to Q\mathcal{C}$ is a fibred functor with base change f^*. The assignment $c \mapsto \mathcal{E}_c$ gives a contravariant functor $Q\mathcal{C} \to \underline{Cat}$.

Recall S acts on $Ex(\mathcal{C})$ by inclusion $S \simeq \mathcal{E}_0 \hookrightarrow Ex(\mathcal{C})$, and $\varphi(a \oplus E) = \varphi(E)$. So we get an induced functor.

$$\Phi := S^{-1}\varphi : S^{-1}(Ex\,\mathcal{C}) \to Q\mathcal{C}.$$

where the fibre over 0 is $S^{-1}S$.

Prop. $\mathcal{C}$ split exact, $S = Iso(\mathcal{C})$, then

$$B(S^{-1}S) \to BS^{-1}(Ex\,\mathcal{C}) \xrightarrow{B\Phi} BQ\mathcal{C}$$

is a homotopy fibration.

Recall Quillen theorem B: $F: \mathcal{C} \to \mathcal{D}$ functor s.t. $\forall f \in Mor(\mathcal{D})$, $f: d \to d'$, the base change functor $f: d'\backslash F \to d\backslash F$ is a homotopy equivalence, then $\forall d \in ob(\mathcal{D})$, the induced sequence $B(d\backslash F) \to B\mathcal{C} \xrightarrow{BF} B\mathcal{D}$ is a homotopy fibration.

We can show Φ is a fibred functor. by using a simple fact.

T Let S acts on X, $F: Y \to X$ is a functor coequalizing the diagram $S \times X \underset{\pi_X}{\overset{\oplus}{\rightrightarrows}} X$ where $\pi_X : S \times X \to X$ is the projection. Assume the base change functor of F

commutes with the action of S on the fibres of F. Then
F ~~is~~ fibred $\Rightarrow$ $S^{-1}F$ also fibred.
f^* homotopy equivalence: easy. suffices to check $f = [0 \hookrightarrow C]$
or $f = [C \twoheadrightarrow 0]$. $\square$

Proof of "$+ = Q$": (sketch). It suffices to show $S^{-1}\mathrm{Ex}(\mathcal{C})$ is contractible.

1). Consider $(Q\mathcal{C})^{mon} \subset Q\mathcal{C}$ be the subcategory with the same objects and morphisms are represented by admissible monics. Note $\mathrm{Ex}(\mathcal{C}) \simeq \mathrm{Sub}[(Q\mathcal{C})^{mon}]$ where Sub denote the Segal subdivision.

<u>Note:</u> Segal subdivision of a category $\mathcal{A}$.

$\mathrm{Ob}(\mathrm{Sub}(\mathcal{A})) = \mathrm{Mor}(\mathcal{A})$ $\quad \mathrm{Mor}_{\mathrm{Sub}(\mathcal{A})}(a \xrightarrow{f} b, a' \xrightarrow{f'} b')$

$= \{(\alpha, \beta) \mid \alpha : a' \to a,\ \beta : b \to b',\ \text{s.t. } f' = \beta f \alpha\}$.

Thus. $\mathrm{Ex}(\mathcal{C}) \simeq \mathrm{Sub}[(Q\mathcal{C})^{mon}] \simeq (Q\mathcal{C})^{mon} \simeq *$.

2) Any action of S on a contractible category is invertible. If all translations are faithful, then $X \to S^{-1}X$, $x \mapsto (s, x)$ are homotopy equivalences $\forall\, s \in \mathrm{Ob}(S)$. Hence.

$$\mathrm{Ex}(\mathcal{C}) \simeq S^{-1}(\mathrm{Ex}\,\mathcal{C}). \quad \square$$

<u>Delooping K-theory.</u>

In each of the construction of $K(A)$, we started from construct some obvious category $\tilde{K}(A)$, and then make some modification, which changes the homotopy property of $B\tilde{K}(A)$.

The comparison between two constructions suggest to define K-theory as functors from rings to spectra.

Recall $\Sigma X = X \wedge S^1$, $\Omega X = \mathrm{Hom}_{Top_*}(S^1, X)$, we have
$[\Sigma X, Y] \simeq [X, \Omega Y]$.

Defn a ~~set~~ spectrum $\mathbb{X} = \{X_0, X_1, \dots\}$ of pointed topological spaces w/ bonding maps $\sigma_i : \Sigma X_i \to X_{i+1}$ $\forall i \geq 0$.

Ex. $\forall X \in Top_*$, define the suspension spectra with $\sigma_i = id$ $\forall i$.
Then we have a functor Σ^∞: Spaces $\to$ Spectra.

Def: An Ω-spectrum ~~X~~ $\mathbb{X}$ is a spectrum s.t. the adjoint bonding maps $\sigma_i^X : X_i \simeq \Omega X_{i+1}$ is a homotopy equivalence.

If a space X is the 0-th component of a Ω-spectrum, then we call it a infinite loop space.
The process of constructing X_i, $i \geq 1$ is the delooping.

Recall $K(A) = K_0 \times BGL(A)^+ =: X_0$.
Q-construction gives $BQ\,\mathbb{P}(A) = X_1$
"$+ = Q$" tells $X_0 \simeq \Omega X_1$.
Q: Is $X_0 = K(A)$ actually the 0-th term of a spectrum?

THM There is a sequence of spaces starting with $X_0 = K(A)$, and $K_1 = BQ(\mathbb{P}(A))$ ~~forgi~~ forming an Ω-spectrum
$\mathbb{K}(A)$ $\qquad$ $\mathbb{K}$: Rings $\to$ Spectra.

Topological categories

Let $C = C^{top}$ be a topological category (i.e. $Ob(C)$, $Mor(C)$ form topological spaces). $\Rightarrow$ $N^{\cdot}C^{top}$ is a simplicial topological space. $\Rightarrow$ $BC^{top} = |N^{\cdot}C^{top}|$ has same underlying set as BC^{δ} (δ means "discrete"/no topology), but the topology of BC^{top} is more intricate: identity may be view as a continuous functor $C^{\delta} \to C^{top}$ $\Rightarrow$ induces a continuous map $BC^{\delta} \to BC^{top}$.

Ex. $G = G^{top}$ be a topological group $\Rightarrow$ two connected spaces BG^{δ} and BG^{top}. Note BG^{δ} has only one nontrivial homotopy group $\pi_1(BG^{\delta}) = G^{\delta}$. In contrast, $\Omega B(G^{top}) \simeq G^{top}$ so $\pi_i BG^{top} = \pi_{i-1} G^{top}$ $i > 0$.

Ex. Let $G = (\mathbb{R}, +)$. $B\mathbb{R}^{top}$ is contractible since $\mathbb{R}^{top}$ is. But $B\mathbb{R}^{\delta}$ is not since $\pi_1(B\mathbb{R}^{\delta}) = \mathbb{R}$.

Higher topological K-theory.

X: paracompact topological space. $VB_{\mathbb{C}}(X)$ is a unital commutative semi-ring.

$$K^0 : Ho(\underline{Top})^{op} \longrightarrow \underline{ComRings}.$$

Fix a vector bundle E on X, we have the dimension function $X \longrightarrow \mathbb{N}$; $x \mapsto \dim E_x$ which is locally constant.

Hence we have a semi-ring homomorphism

$$\dim : VB_{\mathbb{C}}(X) \to [X, \mathbb{N}]$$

$\to$ extends to a unique ring homomorphism

$$\underline{\dim} : K^0(X) \to [X, \mathbb{Z}] = H^0(X, \mathbb{Z})$$

Let $\operatorname{Ker} \underline{\dim} = \widetilde{K}^0(X)$ then we have a natural decomposition

$$K^0(X) \simeq \widetilde{K^0(X)} \oplus H^0(X, \mathbb{Z}).$$

$\widetilde{K_0}$ is representable by $BGL(\mathbb{C})$ as a topological group.

$\underline{\text{THM}}$. $\forall$ ~~cpt~~ compact X

$$K^0(X) \simeq [X, \mathbb{Z} \times BGL(\mathbb{C})^{Top}]$$
$$\widetilde{K}^0(X) \simeq [X, BGL(\mathbb{C})^{Top}]$$

in particular, $\pi_n(BGL(\mathbb{C})^{top}) \simeq \widetilde{K^0}(S^n)$

$\underline{\text{Rmk}}$. 1. $U \hookrightarrow GL_n(\mathbb{C})$ is a deformation retract $\Rightarrow BU \simeq BGL(\mathbb{C})$

2. $K^0(S^d) = \mathbb{Z} \oplus \widetilde{K}^0(S^d) = \begin{cases} \mathbb{Z} & d \equiv 1 \mod 2 \\ \mathbb{Z} \oplus \mathbb{Z} & d \equiv 0 \mod 2. \end{cases}$

so for $n \geq 1$ $\pi_n(BGL(\mathbb{C})^{Top}) = \begin{cases} 0 & n \equiv 1 \mod 2 \\ \mathbb{Z} & n \equiv 0 \mod 2. \end{cases}$

Bott periodicity

higher K-groups.

$$K^{-n} := K^0(\Sigma^n X) \simeq [\Sigma^n X, \mathbb{Z}\times BU]$$
$$\simeq [X, \Omega^n(\mathbb{Z}\times BU)]$$

Note: if $X = *$ then $K^{-n}(*) = [\Sigma^n(*), \mathbb{Z}\times BU]$

$$= [S^n, \mathbb{Z}\times BU] = \pi_n(\mathbb{Z}\times BU)$$
$$\Rightarrow \tilde{K}^{-n}(*) = \begin{cases} \mathbb{Z} & \text{even} \\ 0 & \text{odd} \end{cases}$$

THM (Refined Bott periodicity).

$\exists$ natural homotopy equivalence $\Omega U \simeq \mathbb{Z}\times BU$. In particular $\Omega^2 U \simeq U$.

Since $\Omega BU \simeq U$

$$\Omega^2(\mathbb{Z}\times BU) \overset{\Omega\mathbb{Z}=*}{\simeq} \Omega^2 BU \simeq \Omega U \simeq \mathbb{Z}\times BU.$$

Hence we have the 2-periodicity of the complex topological K-theory $K^{-n-2}(X) = K^{-n}(X)$ $\forall\, n\in\mathbb{Z}$.

Real topological K-theory

$$\Omega^7 O \simeq \mathbb{Z}\times BO \Rightarrow \Omega^8(\mathbb{Z}\times BO) \simeq \mathbb{Z}\times BO.$$
$$\Omega^8 O \simeq O$$